where δ_n is defined by equation (65) of Wait [1]. It is easily found that for $|\tau_e a| \ll 1$:

$$\delta_n = -\frac{in\omega_N^2}{\omega(\nu + i\omega)} \cdot \frac{1}{\tau_p a} \cdot \frac{K_n(\tau_p a)}{K_n'(\tau_p a)} \cdot \quad (4)$$

In (4), n is an integer, K_n is a modified Bessel function, and a dash denotes differentiation with respect to the argument. Equations (3) and (4) describe the low-frequency resonances of the cylindrical cavity. They may be compared with corresponding expressions which have been derived for a spherical cavity [2].

For an incompressible plasma $u=0$. Then τ_p is infinite and $\delta_n=0$. In this case, the resonance condition (3) becomes $\epsilon/\epsilon_0 = -1$ [1], [3]. Budden [3] has considered the low-frequency resonances of a cylindrical cavity in an incompressible anisotropic loss-free plasma. The effect of losses in this situation has also been considered [4].

For $|z| \gg 1$ and $|z| \gg |\nu|$, the modified Bessel function $K_\nu(z)$ of complex order ν and complex argument z satisfies

$$K_\nu(z) \sim (\pi/2z)^{1/2} e^{-z}, \quad -\pi < \arg z < \pi/2 \quad (5)$$

[5], so that $K_\nu'(z)/K_\nu(z) \sim -1$. Hence, for $|\tau_p a| \gg 1$ and $|\tau_p a| \gg |n|$, using (2) and (1), (4) becomes

$$\delta_n \doteqdot \frac{n\omega_N^2}{\omega^2(\nu + i\omega)} \frac{u}{a} \left[1 - \frac{\omega_N^2}{\omega^2} - \frac{i\nu}{\omega}\right]^{-1/2} . \quad (6)$$

This is applicable when the dimensions of the cavity are much greater than the wavelength of the electron acoustic waves in the plasma.

A low-frequency resonance when the plasma is incompressible and loss-free will satisfy (3) with $\delta_n=0$ and $\nu=0$. The only such frequency is $\omega_N/2^{1/2}$ [3]. Thus, for a given electron density, there is only a single resonant frequency, and this frequency is independent of the radius of the cavity. This result may be compared to that for a spherical cavity in an incompressible and loss-free plasma, for which there is an infinite number of resonances [2], [3].

When ν/ω is small and $|\tau_p a|$ is large, the effects of losses and compressibility will be small. The resonant frequencies can then be regarded as having been slightly perturbed from that in the incompressible loss-free case.

Let $(\omega_N/2^{1/2}) + \Omega$, where $|\Omega| \ll \omega_N/2^{1/2}$, be a (complex) resonant frequency when the effects of losses and compressibility are small. When $\nu/\omega \ll 1$, (1) becomes

$$\frac{\epsilon}{\epsilon_0} \doteqdot 1 - \frac{\omega_N^2}{\omega^2} - \frac{i\nu\omega_N^2}{\omega^3} \cdot \quad (7)$$

Neglecting the product $w\nu$ in (6) gives

$$\delta_n \doteqdot \frac{n\omega_N^2}{i\omega^3} \frac{u}{a} \left(1 - \frac{\omega_N^2}{\omega^2}\right)^{-1/2} . \quad (8)$$

Hence, the resonance condition (3) can be written

$$2 - \frac{\omega_N^2}{\omega^2} \doteqdot \frac{i\nu\omega_N^2}{\omega^3} + \delta_n \quad (9)$$

where δ_n is given by (8). Replacing ω in the right-hand side of this expression by its unperturbed resonant value $\omega_N/2^{1/2}$, and using the condition $|\Omega| \ll \omega_N/2^{1/2}$ in the left-hand side, gives

$$\Omega = \frac{i\nu}{2} - \frac{nu}{2a} \cdot \quad (10)$$

Hence, in a shock-excited resonance of the cavity the fields vary with time as

$$\exp\left[i\left(\frac{\omega_N}{2^{1/2}} - \frac{nu}{2a}\right)t\right] \cdot \exp\left(-\frac{\nu}{2}t\right) . \quad (11)$$

Thus, the time constant with which the fields decay depends only on the electron collision frequency; it is independent of the compressibility and the mode number n. This time constant is the same as that found for a spherical cavity in an isotropic slightly lossy, slightly compressible plasma [2].

The fields oscillate with a real frequency which is independent of the losses. The effect of a non-zero value of u is to split the unperturbed resonant frequency $\omega_N/2^{1/2}$ into a series of resonant frequencies, each separated from the next by the amount $u/(2a)$. Of course, for sufficiently large $|n|$, the quantity $|n|u/(2a)$ will no longer be small compared with $\omega_N/2^{1/2}$; then (11) will not be applicable.

R. Burman
Dept. of Physics
Victoria University of Wellington
Wellington, New Zealand

References

[1] J. R. Wait, "Radiation from sources immersed in compressible plasma media," *Can. J. Phys.*, vol. 42, pp. 1760–1780, September 1964.
[2] J. R. Wait, "Resonances of a spherical void in a compressible isotropic plasma," *J. Geophys. Res.*, vol. 69, pp. 4189–4191, October 1, 1964.
[3] K. G. Budden, "The resonant frequencies of cavities in a magnetoionic medium," *Can. J. Phys.*, vol. 42, pp. 90–102, January 1964.
[4] R. Burman, "Resonances of a cylindrical cavity in a lossy magneto-ionic medium," *Proc. IEEE (Correspondence)*, vol. 53, pp. 1147–1148, August 1965.
[5] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics*. New York: Chelsea, 1949, pp. 19 and 22.

Attenuation Constants of Waveguides with General Cross Sections

I. Theory

The cutoff frequencies and the fileld configurations of waveguides with general cross section can be calculated approximately by the point-matching method,[1] provided that the method is applicable. With the field configurations of the ideal waveguide (with perfectly conducting guide walls) known, it is expected that the attenuation constant due to the finite conductivity of the guide walls may be estimated numerically.

Conventionally, the attenuation constant is defined as

$$\alpha = P_L/2P_T \quad (1)$$

if the guide is made of good conducting material, where P_L is the power loss per unit length. The power transfer P_T is given as[2]

$$P_T = (1/2) \int_S \mathrm{Re}\,[\bar{E}_t \times \bar{H}_t^* \cdot \bar{z}]\, dS \quad (2)$$

where S is the cross-sectional area of the waveguide, $\bar{z}$ is the unit vector in the propagating direction, and (*) denotes the operation of taking the complex conjugate. The transverse components of the field E_t and H_t can be calculated from the longitudinal component ψ ($\psi = H_z$ for TE modes, $\psi = E_z$ for TM modes), which was obtained by the point-matching method.[1] Substituting the expressions of E_t and H_t in terms of ψ into (2), and after some manipulation, the power transfer can be reduced to

$$P_T = G \int_S |\psi|^2 dS \quad (3)$$

where

$$G = (1/2Z_0)(f/f_c)^2\zeta \quad \text{for TM modes}$$
$$G = (Z_0/2)(f/f_c)^2\zeta \quad \text{for TE modes}$$
$$\zeta = \sqrt{1 - (f_c/f)^2}$$

and $Z_0 = \sqrt{\mu_0/\epsilon_0}'$ is the intrinsic impedance of free space. The quantities f_c and f are the cutoff and operating frequencies, respectively.

The power loss per unit length of the guide is conventionally estimated by

$$P_L = (R_s/2) \oint_C |H_{\tan}|^2 dl \quad (4)$$

where $R_s = \sqrt{\omega\mu_0/2\sigma}$ is the surface resistance of the guide wall and σ is the conductivity of the conducting material. The path C of the line integral is the contour of the cross section. The integrand in (4) is the square of the magnitude of the magnetic field component tangential to the periphery of the ideal guide walls. *Since the normal component of the transverse magnetic field H_t automatically vanishes at the guide surface*, it is then possible to express $H_{\tan}$ for TM wave modes as follows:

$$|H_{\tan}|^2 = |\bar{H}_t(r_c, \theta)|^2 \quad (5)$$

where r_c, a function of θ, describes the cross-sectional contour. For TE wave modes, however, the longitudinal component of the magnetic field also contributes to the tangential component. Hence,

$$|H_{\tan}|^2 = |\bar{H}_t(r_c, \theta)|^2 + |\psi(r_c, \theta)|^2 . \quad (6)$$

The square of the magnitude of the transverse magnetic field may be written as

$$|\bar{H}_t|^2 = (f/f_c)^2 F(r, \theta) \quad \text{for TM modes} \quad (7)$$

and

$$|\bar{H}_t|^2 = (f\zeta/f_c k^2)^2 F(r, \theta) \quad \text{for TE modes} \quad (8)$$

where

$$F(r, \theta) = \left(\frac{\partial\psi}{\partial r}\right)^2 + \left(\frac{1}{r}\frac{\partial\psi}{\partial\theta}\right)^2 .$$

Combining (1) through (8) yields the following attenuation constants:

$$\alpha = \left(R_s/2Z_0\zeta k^2 \int_S |\psi|^2 dS\right) \oint_C F(r_c, \theta) r_c\, d\theta \quad (9)$$

Manuscript received February 2, 1966. This research work was supported by the National Aeronautics and Space Administration partially funded under NsG-381.

[1] H. Y. Yee and N. F. Audeh, "Uniform waveguides with arbitrary cross-section considered by the point-matching method," *IEEE Transactions on Microwave Theory and Techniques*, vol. MTT-13, pp. 847–851, November 1965.

[2] S. Ramo and J. R. Whinnery, *Fields and Waves in Modern Radio*, 2nd ed. New York: Wiley, 1953, pp. 351–352.

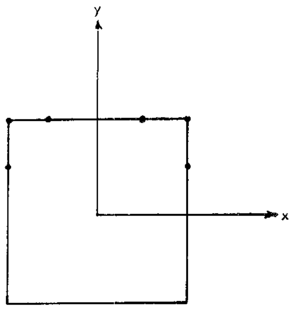

Fig. 1. The square guide with the six chosen points.

prising, because H_z for TE_{10} is antisymmetric with respect to the y-axis. Equation (11) is a good approximation when compared with the exact solution

$$\psi = 0.5 \sin x$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} J_{2n-1}(x) \cos (2n-1)\theta \quad (12)$$

since $J_7(kr)/J_1(kr) < 0.001$ for the largest value of r which is $\sqrt{2}\, a$.

The power transported in the waveguide was calculated numerically using (3) and

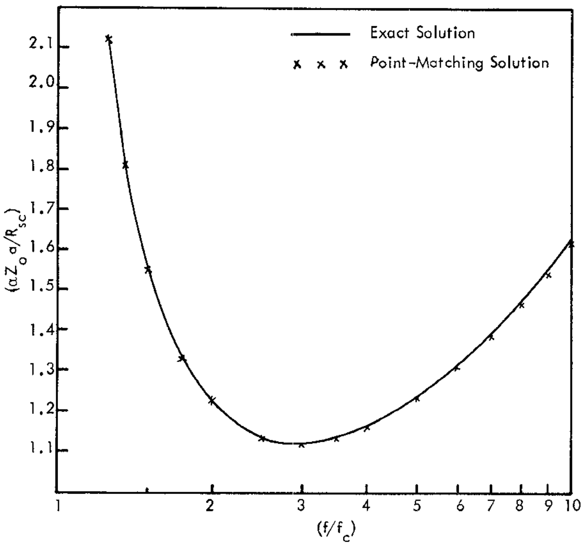

Fig. 2. The attenuation constants of the square waveguide.

for TM wave modes, and

$$\alpha = \left(R_s/2Z_0\zeta \int_S |\psi|^2 dS \right) \left[(\zeta/k)^2 \oint_C F(r_c, \theta) r_c \, d\theta + (f_c/f)^2 \oint_C |\psi(r_c, \theta)|^2 r_c \, d\theta \right] \quad (10)$$

for TE wave modes. The integrations in (9) and (10) can be performed numerically.

II. Numerical Example

To demonstrate the validity of the point-matching method for determination of the field distribution, power transfer, and the attenuation constant, it is assumed that an electromagnetic wave is propagating inside a square waveguide in the TE_{10} mode. The guide has a width of $2a$ and is placed with its center at the origin of a rectangular coordinate system as illustrated in Fig. 1. Since the longitudinal field component H_z is symmetrical with respect to the x-axis for TE_{10}, the sine terms in (1) are omitted. The cutoff wave number calculated by using six points only on the upper half of the guide's cross-sectional contour is 1.5716, compared with the exact value of 1.5708. The expansion coefficients were determined in terms of the coefficient A_1, which is equal to a preassigned value of unity. The resulting wave function is therefore expressed in the following form:

$$\psi = H_z = \sum_{n=1}^{3} (-1)^{n+1} J_{2n-1}(kr) \cos (2n-1)\theta \quad (11)$$

with three-place accuracy. The disappearance of the even terms in (11) is not surprising, because H_z for TE_{10} is antisymmetric with respect to the y-axis.

(11), and the result was 0.1256, which is excellent when compared with the exact value of 0.1250.

The attenuation constant is obtained by substituting the approximate wave function (11) into (10) and performing numerical integrations. It is

$$(\alpha_p a Z_0/R_{sc}) = (0.994/2\zeta)\sqrt{f/f_c}\,[1 + 2.014(f_c/f)^2] \quad (13)$$

where $R_{sc} = R_s\sqrt{f_c/f}$, the surface resistance at cutoff frequency, and α_p denotes the attenuation constant of the point-matching solution. The exact attenuation constant α_e for a square guide of $2a$ is given by[2]

$$(\alpha_e a Z_0/R_{sc}) = (1/2\zeta)\sqrt{f/f_c}\,[1 + 2(f_c/f)^2]. \quad (14)$$

The comparison of these two attenuation constants is shown in Fig. 2 over the frequency range of $(f/f_c) = 1$ through 10. The validity of the attenuation constant calculated by the point matching solutions is then verified.

H. Y. Yee[3]
Dept. of Electrophysics
Polytechnic Institute of Brooklyn
Brooklyn, N. Y.
N. F. Audeh
University of Alabama Research Inst.
Huntsville, Ala.

[3] On a leave of absence from the University of Alabama Research Institute, Huntsville, Ala.

On Transverse Electromagnetic Wave Propagation in a Cylindrically Stratified Magnetoplasma

This communication is concerned with a second-order ordinary differential equation arising in the theory of electromagnetic waves in a cylindrically stratified, axially magnetized plasma. The equation describes transverse propagation when the wave's magnetic field has only an axial component.

Galejs [1] and Yeh and Rusch [2], [3] have studied the transverse propagation of electromagnetic waves in a cylindrically stratified, axially magnetized plasma. It was found that in a continuously varying plasma, the fields are described by two uncoupled wave equations. Thus, the total field can be expressed as the sum of two partial fields which propagate independently. These may be called E-parallel and H-parallel fields, with the electric and magnetic vectors, respectively, having only axial components.

The differential equation describing E-parallel fields is unaffected by the static magnetic field [3]. The equation describing H-parallel fields was derived [1], [3], and its normal form will be considered here.

Cylindrical polar coordinates (r, ϕ, z) are used, with the static magnetic field in the z direction. The inverse permittivity tensor (ϵ^{-1}) in the magnetoplasma is given by [4], [5]

$$\epsilon_0(\epsilon^{-1}) = \begin{bmatrix} M & -iK & 0 \\ iK & M & 0 \\ 0 & 0 & \epsilon_0/\epsilon'' \end{bmatrix} \quad (1)$$

where ϵ_0 is the permittivity of free space. The quantities M, K, and ϵ'' have been defined by Wait [4], [5]. A time factor $e^{i\omega t}$ is taken where ω is the angular frequency of the fields and t is the time. The plasma is taken to be cylindrically stratified with M, K, and ϵ'' depending on r. The permeability has the free space value μ_0.

For propagation transverse to the imposed field, the fields are independent of z. Consider the case in which the magnetic field of the wave has only a z component H. The variables can be separated by writing

$$H = \sum_{n=-\infty}^{\infty} a_n [rM(r)]^{-1/2} f_n(r) e^{-in\phi} \quad (2)$$

where n is an integer and a_n is independent of the coordinates. It is found from Maxwell's equations that $f_n(r)$ satisfies

$$\frac{d^2 f_n}{dr^2} + I(r) f_n = 0 \quad (3)$$

where

$$I(r) = \frac{k_0^2}{M} + \frac{1}{4}\left(\frac{1}{M}\frac{dM}{dr}\right)^2 - \frac{1}{2M}\frac{d^2 M}{dr^2} - \frac{1}{2rM}\frac{dM}{dr} - \frac{4n^2 - 1}{4r^2} - \frac{n}{rM}\frac{dK}{dr} \quad (4)$$

in which $k_0^2 = \omega^2 \mu_0 \epsilon_0$.

Equations (3) and (4) may be compared with equations describing propagation in a planar stratified magnetoplasma [6]. Comments similar to those made previously [6] will now apply in the cylindrical case.

Manuscript received February 11, 1966; revised March 2, 1966.